

LYAPUNOV TYPE INEQUALITY FOR HYBRID FRACTIONAL DIFFERENTIAL EQUATION WITH PRABHAKAR DERIVATIVE

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Abstract

In this paper Lyapunov type inequality is developed for hybrid fractional boundary value problem involving the prabhakar fractional derivative.

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1 Introduction

In the fractional calculus the various integral inequalities plays very important role in the study of qualitative and quantitative properties of solution of differential and integral equations. The well-known Lyapunov result [8] states that if the boundary value problem

$$\begin{cases} y''(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \quad (1.1)$$

has a nontrivial solution, where $q(t)$ is real and continuous function, then

$$\int_a^b |q(u)| du > \frac{4}{b-a}. \quad (1.2)$$

The study of Lyapunov inequalities for the fractional differential equation depends on a fractional differential operator involved and it was initiated by Ferreira [2], also he derived a Lyapunov-type inequality for Riemann-Liouville fractional boundary value problem

$$\begin{cases} D^\alpha y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \quad (1.3)$$

where D^α is the Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$ and $q(t) : [a, b] \rightarrow \mathbb{R}$ is a continuous function. It has been proved that if (1.3) has a

nontrivial solution then

$$\int_a^b |q(u)| du > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}. \quad (1.4)$$

For $\alpha = 2$ the inequality (1.4) reduces to Lyapunov's classical inequality (1.2). Also, Ferriera in [3] obtained a Lyapunov-type inequality for the Caputo fractional boundary value problem

$$\begin{cases} {}^C D^\alpha y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \quad (1.5)$$

where ${}^C D^\alpha$ is the Caputo fractional derivative of order $1 < \alpha \leq 2$. It has been proved in [3] that if (1.5) has a nontrivial solution then

$$\int_a^b |q(u)| du > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \quad (1.6)$$

For $\alpha = 2$, the inequality (1.6) reduces to Lyapunov's classical inequality (1.2). Also, Jleli and Samet [5, 6] modified the above inequalities for fractional differential equations with mixed boundary conditions.

In [10], Surang Sitho and et.al established Lyapunov type inequalities in two different cases for hybrid fractional boundary value problem

$$\begin{cases} D_a^\alpha \left[\frac{y(t)}{f(t, y(t))} - \sum_{i=1}^n I_a^\beta h_i(t, y(t)) \right] + q(t)y(t) = 0, & t \in (a, b), \\ y(a) = y'(a) = y(b) = 0. \end{cases} \quad (1.7)$$

In (1.7) D_a^α denotes the Riemann-Liouville fractional derivative of order $\alpha \in (2, 3]$ starting from a point a , the function $y \in C([a, b], \mathbb{R})$, $g \in L^1((a, b], \mathbb{R})$, $f \in C^1([a, b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $h_i \in C([a, b] \times \mathbb{R}, \mathbb{R})$, $\forall i = 1, 2, \dots, n$ and I_a^β is β -th order Riemann-Liouville integral such that $\beta \geq \alpha$ with the lower limit at a point a .

Recently, in [1] the author's obtained the result on fractional differential equation using the Prabhakar derivative

$$(\mathbf{D}_{\rho, \mu, \omega, a+}^\gamma y)(t) + q(t)y(t) = 0, \quad a < t < b, \quad 1 < \mu \leq 2, \quad \gamma, \rho, \omega \in \mathbb{R}^+ \quad (1.8)$$

with boundray conditions $y(a) = y(b) = 0$. Where $y \in C[a, b]$ and with the help Green function they obtained Lyapunov inequality for the fractional boundary value problem (1.8)

Motivated by above work, in this paper we consider the following hybrid fractional differential equation involving the Prabhakar fractional derivative

$$\begin{cases} \mathbf{D}_{\rho,\mu,\omega,a+}^{\gamma} \left[\frac{y(t)}{p(t,y(t))} - \sum_{i=1}^n \mathbf{E}_{\rho,\mu,\omega,a+}^{\gamma} h_i(t, y(t)) \right] + q(t)y(t) = 0, & t \in (a, b), \\ y(a) = y(b) = 0. \end{cases} \quad (1.9)$$

In (1.9), $\mathbf{D}_{\rho,\mu,\omega,a+}^{\gamma}$ denotes the Prabhakar derivative of order $\mu \in (1, 2]$, $y \in C([a, b], \mathbb{R})$, $g \in L^1((a, b], \mathbb{R})$, $f \in C^1([a, b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $h_i \in C([a, b] \times \mathbb{R}, \mathbb{R})$, $\forall i = 1, 2, \dots, n$ and $\mathbf{E}_{\rho,\mu,\omega,a+}^{\gamma}$ is the Prabhakar integral of order μ with lower limit at a point a . The Lyapunov type inequality is obtained for it.

2 Preliminaries

Definition 2.1 [9] *The generalized Mittag-Leffler function with three parameters is defined as,*

$$E_{\rho,\mu}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k + \mu) k!}, \quad \gamma, \rho, \mu \in \mathbb{C}, \Re(\rho) > 0, \quad (2.1)$$

where $(\gamma)_k$ is Pochhammer symbol defined by,

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1). \text{ for } k = 1, 2, \dots$$

For $\gamma = 1$, the generalized Mittag-Leffler function (2.1) reduces to the two-parameter Mittag-Leffler function given by

$$E_{\rho,\mu}(z) := E_{\rho,\mu}^1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad \rho, \mu \in \mathbb{C}, \quad \Re(\rho) > 0, \quad (2.2)$$

and for $\mu = \gamma = 1$, this function coincides with the classical Mittag-Leffler function $E_{\rho}(z)$

$$E_{\rho}(z) := E_{\rho,1}^1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)}, \quad \rho \in \mathbb{C}, \quad \Re(\rho) > 0. \quad (2.3)$$

Also, for $\gamma = 0$ we have $E_{\rho,\mu}(z) = \frac{1}{\Gamma(\mu)}$.

Definition 2.2 [4] *Let $f \in L^1[0, b]$, $0 < x < b \leq \infty$, the prabhakar integral operator including generalized Mittag-Leffler function (2.1) is defined as follows*

$$\mathbf{E}_{\rho,\mu,\omega,0+}^{\gamma} f(x) dx = \int_0^x (x-u)^{\mu-1} \mathbf{E}_{\rho,\mu}^{\gamma}(\omega(x-u)^{\rho}) f(u) du, \quad x > 0 \quad (2.4)$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$, with $\Re(\rho), \Re(\mu) > 0$.

If for $\gamma = 0$, the prabhakar integral operator coincides with the Riemann-Liouville fractional integral of order μ ;

$$\mathbf{E}_{\rho,\mu,\omega,0+}^0 f(x) = I_{0+}^{\mu} f(x),$$

where the Riemann-Liouville fractional integral is defined as

$$I_{0+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt, \quad \mu \in \mathbb{C}, \Re(\mu) > 0. \quad (2.5)$$

Definition 2.3 [4] Let $f \in L^1[0, b]$, $0 < x < b \leq \infty$, the Prabhakar derivative is defined as

$$D_{\rho, \mu, \omega, 0+}^{\gamma} f(x) = \frac{d^m}{dx^m} E_{\rho, m-\mu, \omega, 0+}^{-\gamma} f(x), \quad (2.6)$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$, with $\Re(\rho) > 0$, $\Re(\mu) > 0$, $m-1 < \Re(\mu) < m$.

We note that the Prabhakar derivative generalizes the Riemann-Liouville fractional derivative

$$D_{0+}^{\mu} f(x) = \frac{d^m}{dx^m} \left(I_{0+}^{m-\mu} f \right)(x), \quad \mu \in \mathbb{C}, \Re(\mu) > 0, m-1 < \Re(\mu) < m. \quad (2.7)$$

Lemma 2.1 [9] The Laplace transform of generalized Mittag-Leffler function (2.1) is given by

$$\mathcal{L}[x^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega x^{\rho})](s) = s^{-\mu} (1 - \omega s^{-\rho})^{-\gamma}, \quad |\omega s^{-\rho}| < 1, \quad (2.8)$$

for $\gamma, \rho, \mu, \omega, s \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(s) > 0$.

Lemma 2.2 [7] Let $\gamma, \rho, \mu, \omega, s \in \mathbb{C}$ with $\Re(\mu) > 0$. Then for any $n \in \mathbb{N}$ differentiation of the generalized Mittag-Leffler function (2.1) is given by

$$\left(\frac{d}{dx} \right)^n [x^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega x^{\rho})] = x^{\mu-n-1} E_{\rho, \mu-n}^{\gamma}(\omega x^{\rho}). \quad (2.9)$$

Lemma 2.3 [1] The Laplace transform of Prabhakar integral (2.4) is given by

$$\mathcal{L}\{E_{\rho, \mu, \omega, 0+}^{\gamma} f(x); s\} = s^{-\mu} (1 - \omega s^{-\rho})^{-\gamma} F(s), \quad (2.10)$$

where $F(s)$ is the Laplace transform of $f(x)$, and it is written as

$$F(s) = \mathcal{L}\{f(x); s\} = \int_0^{\infty} e^{-sx} f(x) dx, \quad s \in \mathbb{C}. \quad (2.11)$$

Lemma 2.4 [1] The Laplace transform of Prabhakar derivative (2.6) is given by

$$\mathcal{L}\left\{D_{\rho, \mu, \omega, 0+}^{\gamma} f(x); s\right\} = s^{\mu} (1 - \omega s^{-\rho}) F(s) - \sum_{k=1}^{m-1} s^k (D_{\rho, \mu-k-1, \omega, 0+}^{\gamma} f)(0). \quad (2.12)$$

Lemma 2.5 [1] If $f(x) \in C(a, b) \cap L(a, b)$, then

$$\mathbf{D}_{\rho, \mu, \omega, a+}^{\gamma} \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} f(x) = f(x), \quad (2.13)$$

and if

$f(x), \mathbf{D}_{\rho, \mu, \omega, a+}^{\gamma} f(x) \in C(a, b) \cap L(a, b)$ then for $c_j \in \mathbb{R}$, and $m - 1 < \mu \leq m$, we have

$$\begin{aligned} \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} \mathbf{D}_{\rho, \mu, \omega, a+}^{\gamma} f(x) &= f(x) + c_1(x - a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(x - a)^{\rho}) \\ &\quad + c_2(x - a)^{\mu-2} E_{\rho, \mu-1}^{\gamma}(\omega(x - a)^{\rho}) + \dots \\ &\quad + c_m(x - a)^{\mu-m} E_{\rho, \mu-m+1}^{\gamma}(\omega(x - a)^{\rho}). \end{aligned} \quad (2.14)$$

The authors had given the following proved lemma in [1].

Lemma 2.6 The Green function defined by

$$G(t, u) = \begin{cases} \frac{(t-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho})}{(b-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) \\ \quad - (t-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-u)^{\rho}), & a \leq u \leq t \leq b, \\ \frac{(t-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho})}{(b-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}), & a \leq t \leq u \leq b. \end{cases} \quad (2.15)$$

satisfies the following conditions :

1. For all $a \leq t, u \leq b$, $G(t, u) \geq 0$.
2. $\max_{t \in [a, b]} G(t, u) = G(u, u)$, for $u \in [a, b]$.
3. The maximum of $G(u, u)$ is given at $u = \frac{a+b}{2}$ and has value

$$\max_{u \in [a, b]} G(u, u) = G\left(\frac{a+b}{2}, \frac{a+b}{2}\right) = \left(\frac{b-a}{4}\right)^{\mu-1} \frac{E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})}. \quad (2.16)$$

3 Main Results

In this section, we have obtained Lyapunov type inequalities in two different cases:

(I) $h_i(t, y(t)) = 0$, $i = 1, 2, \dots, n$ and

(II) $h_i(t, y(t)) \neq 0$, $i = 1, 2, \dots, n$.

Case I : $h_i(t, y(t)) = 0$, $i = 1, 2, \dots, n$

Here, we consider the problem (1.8) with $h_i(t, y(t)) = 0$, $\forall t \in [a, b]$, and for $\mu \in (1, 2]$.

We first construct a Green function for the following boundary value problem

$$\begin{cases} \mathbf{D}_{\rho, \mu, \omega, a+}^{\gamma} \left[\frac{y(t)}{p(t, y(t))} \right] + q(t)y(t) = 0, & 1 < \mu \leq 2, \quad \gamma, \rho, \omega \in \mathbb{R}^+, \\ y(a) = y(b) = 0. \end{cases} \quad (3.1)$$

Theorem 3.1 *Let $y \in AC([a, b], \mathbb{R})$ be a solution of (3.1). Then the function $y(t)$ satisfies the following integral equation*

$$y(t) = p(t, y(t)) \int_a^b G(t, u) q(u) y(u) du, \quad (3.2)$$

where the Green function $G(t, u)$ is given by (2.15).

Proof: Operating $\mathbf{E}_{\rho, \mu, \omega, a+}^\gamma$ on hybrid fractional differential equation (3.1) and using lemma (2.5) for real constant c_1 and c_2 we have

$$\begin{aligned} \frac{y(t)}{p(t, y(t))} &= c_1(t-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(t-a)^\rho) + c_2(t-a)^{\mu-2} E_{\rho, \mu-1}^\gamma(\omega(t-a)^\rho) - \mathbf{E}_{\rho, \mu, \omega, a+}^\gamma q(t) y(t) \\ &= c_1(t-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(t-a)^\rho) + c_2(t-a)^{\mu-2} E_{\rho, \mu-1}^\gamma(\omega(t-a)^\rho) \\ &\quad - \int_a^t (t-u)^{\mu-1} E_{\rho, \mu-1}^\gamma(\omega(t-a)^\rho) q(y) y(u) du. \end{aligned}$$

Now, by employing the boundary conditions we obtain the value of c_1 and c_2 as follows

$$\begin{aligned} y(a) = 0 &\Leftrightarrow 0 = c_2(t-a)^{\mu-2} E_{\rho, \mu-1}^\gamma(\omega(t-a)^\rho) - \int_a^a (a-u)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(a-u)^\rho) q(u) y(u) du, \\ &\Leftrightarrow 0 = c_2(t-a)^{\mu-2} E_{\rho, \mu-1}^\gamma(\omega(t-a)^\rho), \\ &\Leftrightarrow c_2 = 0, \end{aligned}$$

and

$$\begin{aligned} y(b) = 0 &\Leftrightarrow 0 = c_1(b-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-a)^\rho) - \int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-u)^\rho) q(u) y(u) du, \\ &\Leftrightarrow c_1 = \frac{1}{(b-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-a)^\rho)} \int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-u)^\rho) q(u) y(u) du. \end{aligned}$$

Therefore the unique solution of (3.1) is written as follows

$$\begin{aligned} \frac{y(t)}{p(t, y(t))} &= \int_a^t \left[\frac{(t-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(t-a)^\rho)}{(b-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-a)^\rho)} (b-u)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-u)^\rho) \right. \\ &\quad \left. - (t-u)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(t-u)^\rho) \right] q(u) y(u) du \\ &\quad + \int_t^b \left[\frac{(t-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(t-a)^\rho)}{(b-a)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-a)^\rho)} (b-u)^{\mu-1} E_{\rho, \mu}^\gamma(\omega(b-u)^\rho) \right] q(u) y(u) du, \\ y(t) &= p(t, y(t)) \int_a^b G(t, u) q(u) y(u) du, \end{aligned}$$

where $G(t, u)$ is given by (2.15).

Theorem 3.2 Let $\mathcal{B} = C[a, b]$ be the Banach space equipped with norm $\|y\| = \sup_{t \in [a, b]} |y(t)|$ and nontrivial continuous solution of the hybrid fractional boundary value problem

$$\begin{cases} D_{\rho, \mu, \omega, a+}^{\gamma} \left[\frac{y(t)}{p(t, y(t))} \right] + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases}$$

exists, then

$$\frac{1}{\|p\|} \left(\frac{4}{b-a} \right)^{\mu-1} \frac{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})}{E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})} < \int_a^b |q(u)| du, \quad (3.3)$$

where $q(t)$ is a real and continuous function.

Proof: According to theorem (3.1), a solution of the above fractional boundary value problem satisfies the integral equation

$$y(t) = p(t, y(t)) \int_a^b G(t, u) q(u) y(u) du, \quad (3.4)$$

which by applying the indicated norm on both sides of it, gives

$$\begin{aligned} \|y\| &\leq \|p\| \|y\| \max_{t \in [a, b]} |G(t, u)| \int_a^b |q(u)| du, \\ 1 &\leq \|p\| \max_{t \in [a, b]} |G(t, u)| \int_a^b |q(u)| du. \end{aligned}$$

Using the second property of the Green function in Lemma (2.6), we get desired inequality

$$\begin{aligned} 1 &< \|p\| \left(\frac{b-a}{4} \right)^{\mu-1} \frac{E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} \int_a^b |q(u)| du, \\ \frac{1}{\|p\|} \left(\frac{4}{b-a} \right)^{\mu-1} \frac{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})}{E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})} &< \int_a^b |q(u)| du. \quad \square \end{aligned}$$

Now, we consider the second case.

Case II : $h_i(t, y(t)) \neq 0, \quad i = 1, 2, \dots, n.$

In this case, we construct Lyapunov type inequality for hybrid fractional boundary value problem (1.9).

Theorem 3.3 Let $y \in AC[a, b]$ be a solution of (1.9), then the function $y(t)$ satisfy the following integral equation,

$$y(t) = p(t, y(t)) \int_a^b G(t, y(t)) [y(u)q(u) - \sum_{i=1}^n h_i(u, y(u))] du, \quad (3.5)$$

where $G(t, u)$ is Green function defined as in (2.15).

Proof: Operating Prabhakar integral on (1.8) we get

$$\begin{aligned} \frac{y(t)}{p(t, y(t))} - \sum_{i=1}^n \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} h_i(t, y(t)) + c_1(t-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) \\ + c_2(t-a)^{\mu-2} E_{\rho, \mu-1}^{\gamma}(\omega(t-a)^{\rho}) + \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} q(t)y(t) = 0. \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned} \frac{y(t)}{p(t, y(t))} = \sum_{i=1}^n \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} h_i(t, y(t)) - \int_a^t (t-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-u)^{\rho}) q(u) y(u) du \\ + c_1(t-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) + c_2(t-a)^{\mu-2} E_{\rho, \mu-1}^{\gamma}(\omega(t-a)^{\rho}). \end{aligned} \quad (3.6)$$

Now, by employing the boundary conditions we can obtain the value of coefficients c_1 and c_2 as

$$\begin{aligned} y(a) = 0 &\Leftrightarrow \sum_{i=1}^n \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} h_i(a, y(a)) = c_2(t-a)^{\mu-2} E_{\rho, \mu-1}^{\gamma}(\omega(t-a)^{\rho}), \\ &\Leftrightarrow c_2 = \frac{\sum_{i=1}^n \int_a^a (t-u)^{\mu-1} \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} h_i(u, y(u)) (\omega(t-u)^{\rho}) du}{(t-a)^{\mu-2}} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) \\ &\Leftrightarrow c_2 = 0, \end{aligned}$$

and

$$\begin{aligned} y(b) = 0 &\Leftrightarrow \sum_{i=1}^n \left[\mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} h_i(t, y(t)) \right]_{t=b} = - \int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) q(u) y(u) du \\ &\quad + c_1(b-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho}), \\ &\Leftrightarrow c_1 = \frac{(b-a)^{\mu-1}}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} \left\{ \int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) q(u) y(u) du \right. \\ &\quad \left. - \sum_{i=1}^n \int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) h_i(u, y(u)) du \right\}. \end{aligned}$$

Substituting these value of c_1 and c_2 in equation (3.6) we get

$$\begin{aligned}
y(t) &= p(t, y(t)) \left\{ \sum_{i=1}^n \mathbf{E}_{\rho, \mu, \omega, a+}^{\gamma} h_i(t, y(t)) - \int_a^t (t-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-u)^{\rho}) q(u) y(u) du \right. \\
&\quad + \frac{(b-a)^{1-\mu} (t-a)^{\mu-1}}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) \left[\int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) q(u) y(u) du \right. \\
&\quad \left. \left. - \sum_{i=1}^n \int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) h_i(u, y(u)) du \right] \right\} \\
&= p(t, y(t)) \left\{ \sum_{i=1}^n \int_a^t (t-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-u)^{\rho}) h_i(u, y(u)) du \right. \\
&\quad - \int_a^t (t-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-u)^{\rho}) q(u) y(u) du \\
&\quad + \frac{(b-a)^{1-\mu} (t-a)^{\mu-1}}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) \left[\int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) q(u) y(u) du \right] \\
&\quad \left. - \frac{(b-a)^{1-\mu} (t-a)^{\mu-1}}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) \left[\sum_{i=1}^n \int_a^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) h_i(u, y(u)) du \right] \right\}, \\
y(t) &= p(t, y(t)) \left\{ - \int_a^t (t-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-u)^{\rho}) q(u) y(u) du \right. \\
&\quad + \int_a^t \frac{(t-a)^{\mu-1} (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho})}{(b-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} q(u) y(u) du \\
&\quad + \int_t^b \frac{(t-a)^{\mu-1} (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho})}{(b-a)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} q(u) y(u) du \\
&\quad + \sum_{i=1}^n \int_a^t (t-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(t-u)^{\rho}) h_i(u, y(u)) du \\
&\quad - \frac{(b-a)^{1-\mu} (t-a)^{\mu-1}}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) \sum_{i=1}^n \int_a^t (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) h_i(u, y(u)) du \\
&\quad \left. - \frac{(b-a)^{1-\mu} (t-a)^{\mu-1}}{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})} E_{\rho, \mu}^{\gamma}(\omega(t-a)^{\rho}) \sum_{i=1}^n \int_t^b (b-u)^{\mu-1} E_{\rho, \mu}^{\gamma}(\omega(b-u)^{\rho}) h_i(u, y(u)) du \right\}, \\
y(t) &= p(t, y(t)) \left[\int_a^b G(t, u) q(u) y(u) - \sum_{i=1}^n \int_a^b G(t, u) h_i(u, y(u)) du \right], \\
y(t) &= p(t, y(t)) \int_a^b G(t, u) \left[q(u) y(u) - h_i(u, y(u)) \right] du,
\end{aligned}$$

which is desired result. \square

To prove our next result we use the following condition
 $|q(u)y(u) - \sum_{i=1}^n h_i((u, y(u)))| \leq K|q(u)||y|.$

Theorem 3.4 *Let $\mathcal{B} = C[a, b]$ be the Banach space equipped with norm $\|y\| = \sup_{t \in [a, b]} |y(t)|$, and a nontrivial continuous solution of the hybrid fractional boundary value problem (1.8) exist, then*

$$\frac{1}{K\|p\|} \left(\frac{4}{b-a} \right)^{\mu-1} \frac{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})}{E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})} < \int_a^b |q(u)| du, \quad (3.7)$$

where $q(t)$ is real and continuous function.

Proof: In accordance with theorem (3.3), a solution of the above hybrid fractional boundary value problem (1.8), satisfies the integral equation

$$y(t) = p(t, y(t)) \int_a^b G(t, u) \left[y(u)q(u) - \sum_{i=1}^n h_i(u, y(u)) \right] du,$$

which by applying the indicated norm on both sides of it, gives

$$\begin{aligned} \|y\| &\leq \|p\| \int_a^b |G(t, u)| |y(u)q(u) - \sum_{i=1}^n h_i(u, y(u))| du, \\ \|y\| &\leq \|p\| \max_{t \in [a, b]} |G(t, u)| \int_a^b |q(u)y(u) - \sum_{i=1}^n h_i(u, y(u))| du, \\ \|y\| &\leq K\|p\|\|y\| \max_{t \in [a, b]} |G(t, u)| \int_a^b |q(u)| du. \end{aligned}$$

Using the second property of theorem (2.6), we get the desired inequality

$$\int_a^b |q(u)| du > \frac{1}{K\|p\|} \left(\frac{4}{b-a} \right)^{\mu-1} \frac{E_{\rho, \mu}^{\gamma}(\omega(b-a)^{\rho})}{E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho}) E_{\rho, \mu}^{\gamma}(\omega(\frac{b-a}{2})^{\rho})}. \quad \square$$

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